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Motions at subcritical values of the Rayleigh number in a rotating fluid

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A simple analysis is presented for the finite-amplitude, steady motions in a rotating layer of fluid which is heated uniformly from below and cooled from above. The boundaries are considered to be 'free' and a solution is obtained for the two-dimensional problem using the eigenfunctions of the stability problem plus the smallest number of higher modes required to represent non-linear interactions. In his analysis of the stability problem Chandrasekhar (1953) concluded that in mercury overstable motions can occur for a value of the Rayleigh number which is as little as 1/67 of the value required for instability to steady motions. In the present paper it is shown that, for a restricted range of Taylor number, steady finite-amplitude motions can exist for values of the Rayleigh number smaller than the critical value required for overstability. The horizontal scale of these finite-amplitude steady motions is larger than that of the overstable motions. A more exact solution to the finite-amplitude problem confirms the above results. The latter solution together with additional physical results will be presented in a later paper.

1. Introduction and summary

In an earlier paper, Veronis (1959),† finite-amplitude solutions were derived for a layer of fluid heated from below, cooled from above and subjected to a rotation about the vertical axis. The method which was employed in that paper was a perturbation expansion which was pivoted about the solution to the infinitesimal stability problem. One of the conclusions drawn from the earlier analysis was that in fluids with a small Prandtl number (< 1) motion should exist at values of the Rayleigh number (effectively, the ratio of destabilizing forces to dissipation forces) below the critical value derived from linear stability theory. The results depend strongly on the cellular shape which the convection assumes.

Recently, experiments by L. Koschmieder for the non-rotating system and by H. T. Rossby for a rotating fluid‡ indicate that the preferred cellular pattern of convection consists of two-dimensional rolls whose orientation is determined by the lateral boundaries. The present paper assumes that convective motions do, in fact, occur as a two-dimensional pattern of rolls and the finite-amplitude

† Hereafter referred to as I.
‡ Neither study has yet been published.

problem is analysed for two-dimensional motions in a fluid of small Prandtl number where the boundaries are assumed to be 'slippery' as well as perfectly conducting.

The method of analysis is based on a truncated representation of the finite-amplitude motions. Specifically, the form of the velocity and temperature fields is represented by the marginally stable modes plus the first distortion of these modes by non-linear interaction. No other modes are admitted in the representation. The resulting non-linear equations for the modal amplitudes are then solved on the assumption that the motion is steady. Although such an approach involves a drastic oversimplification of the form of the velocity and temperature fields, especially if the analysis is extended to values of the Rayleigh number far from the critical value, it does represent the simplest non-linear analysis for which results can be obtained.

The next section contains the mathematical formulation of the problem. The equations are non-dimensionalized and the important parameters, the Prandtl number, the Taylor number (and a modified Taylor number), and the Rayleigh number, are introduced.

In §3 a simple physical argument is given for the occurrence of instability to infinitesimal, overstable (as opposed to steady) motions, It is shown there that, when the Prandtl number is sufficiently small, overstable motions decrease the constraining effect of rotation and it is for that reason that convection can exist at a value of the Rayleigh number smaller than that which is required for instability to marginally steady motions. The same argument is valid for the existence of instability to finite-amplitude motions. Hence, the study of finite-amplitude instability is naturally suggested. Also in §3 it is shown that for the overstable problem a more appropriate parameter than the Taylor number is a modified Taylor number, viz. the Taylor number defined with the thermometric diffusivity replacing the kinematic viscosity. This same parameter is a more natural one to use in the finite-amplitude problem.

The final section contains the finite-amplitude analysis using a truncated representation of five components. The method of solution is essentially the Galerkin method, although the purpose here is to derive information about the amplitudes as functions of the eigenvalue parameter. Mathematically, the problem reduces to finding a solution to an algebraic quadratic equation. Physically, however, the results are far from trivial. It is shown that finite-amplitude solutions exist for subcritical values of the Rayleigh number as long as the modified Taylor number is smaller than some maximum value (approximately 1000) and as long as the Prandtl number σ is less than the reciprocal of the horizontal wave number. This range of Taylor number is readily accessible in a laboratory experiment with mercury and the phenomenon should be observable.* The motion involves a cellular pattern with a horizontal spacing which is the same as that of the non-rotating system (larger horizontal dimensions than those predicted by infinitesimal theory).

* *Note added in proof.* In experiments with mercury, H. T. Rossby has observed finite amplitude steady motions at sub-critical values of the Rayleigh number in the range of Taylor numbers suggested by this analysis.

Only this single finite-amplitude result is presented in this paper. Additional information, such as heat flux *vs* Rayleigh number, is deferred to a later paper, where the analysis has been extended to include as many as 85 components. The reason for deferring the results for heat flux is that the heat flux is quantitatively much larger when more components are included. However, the existence of finite-amplitude motions at subcritical values of the Rayleigh number has been confirmed by the study with many more components.

2. Mathematical formulation

The lower boundary ($z = 0$) of the layer of fluid is maintained at temperature T_0 and the temperature of the upper boundary ($z = d$) is $T_0 - \Delta T$. We write the total temperature as

$$T_{\text{total}} = T_0 - \Delta T(z/d) + T(x, z, t), \tag{2.1}$$

where $T(x, z, t)$ is the deviation of the temperature from the linear profile. In contrast to the formulation in I the deviation, $T(x, z, t)$, of the temperature contains a non-zero horizontal mean.

Then the equations (all variations with respect to y are assumed to vanish) are the two-dimensional Boussinesq equations for the conservation of momentum

$$\partial \mathbf{v}/\partial t + \mathbf{v} . \nabla \mathbf{v} + 2\mathbf{\Omega} \times \mathbf{v} = -\rho_0^{-1} \nabla p - \mathbf{g}\rho'/\rho_0 + \nu \nabla^2 \mathbf{v}, \tag{2.2}$$

the conservation of mass

$$\partial u/\partial x + \partial w/\partial z = 0, \tag{2.3}$$

the linear equation of state for the fluctuation density

$$\rho' = -\rho_0 \alpha T, \tag{2.4}$$

and the equation for the conservation of heat

$$\partial T/\partial t - w(\Delta T/d) + \mathbf{v} . \nabla T = \kappa \nabla^2 T. \tag{2.5}$$

Here, $\mathbf{v}$ is the three-dimensional velocity vector with components (u, v, w) in the respective directions (x, y, z); $\mathbf{\Omega}$ is the constant rate of rotation of the entire system about the vertical (z) axis; $\mathbf{g}$ is the gravitational acceleration in the negative z-direction; ρ_0 is the density at temperature T_0; α is the coefficient of thermal expansion; and ν and κ are respectively the coefficients of kinematic viscosity and thermometric diffusivity. In equation (2.5) the linear part of T_{total} has been separated out and appears as the second term.

We cross-differentiate the first and third equations of motion in order to eliminate the pressure p. Then defining the y-component of vorticity

$$\eta = \partial u/\partial z - \partial w/\partial x, \tag{2.6}$$

we have

$$\partial \eta/\partial t + \mathbf{v} . \nabla \eta - 2\Omega(\partial v/\partial z) = -g\alpha(\partial T/\partial x) + \nu \nabla^2 \eta. \tag{2.7}$$

The second equation of motion has the form

$$\partial v/\partial t + \mathbf{v} . \nabla v + 2\Omega u = \nu \nabla^2 v. \tag{2.8}$$

We introduce the stream function, ψ, through the definitions

$$u = \partial \psi/\partial z, \quad w = -\partial \psi/\partial x \tag{2.9}$$

so that

$$\eta = \partial u/\partial z - \partial w/\partial x = \nabla^2 \psi. \tag{2.10}$$

Our system then becomes

$$\frac{\partial \eta}{\partial t} = J(\psi, \eta) + 2\Omega \frac{\partial v}{\partial z} - g\alpha \frac{\partial T}{\partial x} + \nu \nabla^2 \eta, \tag{2.11}$$

$$\frac{\partial v}{\partial t} = J(\psi, v) - 2\Omega \frac{\partial \psi}{\partial z} + \nu \nabla^2 v, \tag{2.12}$$

$$\frac{\partial T}{\partial t} = J(\psi, T) - \frac{\Delta T}{d}\frac{\partial \psi}{\partial x} + \kappa \nabla^2 T, \tag{2.13}$$

where J stands for the Jacobian. Furthermore, the system is non-dimensionalized by

$$\mathbf{v} = (\kappa/d)\,\mathbf{v}', \quad t = (d^2/\kappa)\,t', \quad (x, z) = d(x', z'), \quad T = (\Delta T)\,T', \tag{2.14}$$

where the primed quantities are non-dimensional. Then equations (2.11) to (2.13) become

$$\partial\eta/\partial t = J(\psi, \eta) + \sigma\mathscr{T}(\partial v/\partial z) - \sigma R(\partial T/\partial x) + \sigma\nabla^2\eta, \tag{2.15}$$

$$\partial v/\partial t = J(\psi, v) - \sigma\mathscr{T}(\partial\psi/\partial z) + \sigma\nabla^2 v, \tag{2.16}$$

$$\partial T/\partial t = J(\psi, T) - \partial\psi/\partial x + \nabla^2 T, \tag{2.17}$$

where all of the variables are now non-dimensional, the primes have been dropped and the following non-dimensional parameters appear:

$$\left.\begin{array}{ll} \text{Prandtl number,} & \sigma = \nu/\kappa; \\ \text{Taylor number,} & \mathscr{T}^2 = 4\Omega^2 d^4/\nu^2; \\ \text{Rayleigh number,} & R = g\alpha\Delta T d^3/\kappa\nu. \end{array}\right\} \tag{2.18}$$

It will be noted that the Prandtl number and the square root of the Taylor number appear in product form as

$$\mathscr{S} = \sigma\mathscr{T} = 2\Omega d^2/\kappa, \tag{2.19}$$

so that we could equally well have defined this combination (i.e. κ replacing ν in the Taylor number) as the non-dimensional number containing the rotational parameter. In fact for the convection problem $\mathscr{S}$ is more appropriate than the Taylor number. Only in the steady, linear stability problem does $\mathscr{T}$ appear independently of the Prandtl number.

The boundary conditions for the problem are straightforward. With the boundaries at $z = 0$ and $z = 1$ taken as flat, stress-free boundaries and as perfect conductors the conditions become

$$\psi = 0, \quad \partial^2\psi/\partial z^2 = 0, \quad T = 0, \quad \partial v/\partial z = 0 \quad \text{on} \quad z = 0,\ 1. \tag{2.20}$$

3. Simple physical arguments about the roles of time-dependence and non-linearity

3.1. *Summary of stability results*

The formal solution to the stability problem was first given by Chandrasekhar (1953) and is also discussed in I. We summarize the results briefly.

The system is marginally stable to infinitesimal perturbations of the form

$$\psi \sim \sin\pi\alpha x \sin n\pi z, \quad T \sim \cos\pi\alpha x \sin n\pi z, \quad v \sim \sin\pi\alpha x \cos n\pi z \tag{3.1}$$

(where α is the horizontal wave number) for a critical value of the Rayleigh number given by

$$R_c = [(n^2+\alpha^2)^3\pi^4 + n^2\mathscr{T}^2]/\alpha^2. \tag{3.2}$$

For a given value of $\mathscr{T}$ the minimum value of R_c for instability occurs for $n = 1$ and for α^2 given by

$$2\alpha^6 + 3\alpha^4 = 1 + \mathscr{T}^2/\pi^4. \tag{3.3}$$

Thus for $\mathscr{T}^2 \gg \pi^4$, we have

$$\alpha^2 \to (\mathscr{T}^2/2\pi^4)^{\frac{1}{3}}, \quad R_c^{\min} \to \tfrac{3}{2}(2\pi^4)^{\frac{1}{3}}\,\mathscr{T}^{\frac{4}{3}}. \tag{3.4}$$

If the marginally stable motions are allowed to depend on time (overstable motions), i.e. if ψ, v and T are also proportional to e^{ipt}, where p is real, then the critical value of the Rayleigh number, now denoted by R_0, is

$$R_0 = 2(\sigma+1)\,\alpha^{-2}[(\alpha^2+n^2)^3\pi^4 + n^2\mathscr{S}^2/(\sigma+1)^2], \tag{3.5}$$

which has a minimum for $n = 1$ and α^2 given by

$$2\alpha^6 + 3\alpha^4 = 1 + \mathscr{S}^2/\pi^4(\sigma+1)^2. \tag{3.6}$$

Thus when $\mathscr{S}^2 \gg \pi^4$

$$\alpha^2 \to [\mathscr{S}^2/2(\sigma+1)^2\pi^4]^{\frac{1}{3}}, \quad R_0^{\min} \to (2\pi^4)^{\frac{1}{3}}\,3(\sigma+1)\,[\mathscr{S}^2/(\sigma+1)^2]^{\frac{2}{3}}. \tag{3.7}$$

The usefulness of $\mathscr{S}$ for overstable motions is clear from the asymptotic expression for $R_0^{\min}$. As σ becomes very small, $R_0^{\min} \to \text{const}\,\mathscr{S}^{\frac{4}{3}}$; hence we have a single asymptotic curve for large $\mathscr{S}$. These overstable motions can exist only for $\sigma < 1$. Associated with these values for α^2 and R_0 is a value of p, the frequency, given by

$$p^2 = -\sigma^2\pi^4(\alpha^2+1)^2 + \mathscr{S}^2(1-\sigma)/(\alpha^2+1)(1+\sigma). \tag{3.8}$$

Since $p^2 > 0$, a necessary condition for the existence of overstable motions is

$$\mathscr{S}^2 \geqslant \sigma^2(\alpha^2+1)^3(\sigma+1)/(1-\sigma). \tag{3.9}$$

Mercury with a Prandtl number of 0·025 first becomes unstable to overstable motions. For mercury the ratio of R_c to R_0 for large values of $\mathscr{T}$ is

$$R_c/R_0 \to (1+\sigma)^{\frac{1}{3}}/2\sigma^{\frac{4}{3}} \approx 67,$$

i.e. the temperature difference required for marginal instability to infinitesimal steady convective motions is about 67 times that required for marginal instability to infinitesimal time-dependent motions. Hence, we would expect the instability in mercury to occur as overstable motions provided that (3.9) is satisfied.

3.2. *Why overstable motions occur*

Even though the analysis by Chandrasekhar shows that overstable motions will occur for a fluid with a sufficiently small Prandtl number if it is rotated at a sufficiently high rate, we may still ask for a simple physical argument to explain the result. An attempt to present such an argument is given below.

In the non-rotating stability problem the horizontal temperature gradient of the perturbed field releases potential energy and the latter is balanced by the viscous dissipation of the motion. Thermally the upward convection of warm fluid is balanced by the diffusion of the excess temperature. In these simple balances the motion and temperature fields are in phase and no restoring force exists; hence no time-dependent motions are possible.

In the steady rotating system the rotation introduces a Coriolis force and a 'thermal wind' component is generated. The thermal wind field is a familiar concept in geophysical problems—it describes a balance between a horizontal temperature gradient and the vertical shear of the velocity component (we shall call it the zonal velocity) normal to the temperature gradient. In equation (2.15) this balance is given by the terms $\sigma\mathscr{T}(\partial v/\partial z)$ and $\sigma R(\partial T/\partial x)$. Of course, the balance is not complete because a third force, dissipation, is also present. However, the inhibition of convection by rotation is clearly traceable to the thermal wind because a good part of the force which releases potential energy is now balanced by the rotational constraint which is energetically inactive. The larger the rotation rate, the larger the zonal velocity. Hence, less potential energy is released for a given horizontal temperature gradient. Because a component of motion exists parallel to the boundaries of the roll, the boundaries of the cell are closer together so that the viscous dissipation is increased. Furthermore, the zonal velocity component introduces additional dissipation and this is balanced by the Coriolis term, $\sigma\mathscr{T}u$. Thus, part of the circulation velocity, u, is now taken up to balance the dissipation of zonal velocity which was created by the rotation of the system. The thermal balance is unaffected in the steady problem and the upward (downward) motion is still completely in phase with warmer (colder) fluid.

Time-dependent motions of various types can exist in a rotating fluid because the Coriolis forces can act as a restoring mechanism; e.g. in a horizontally infinite fluid one type of oscillation which can exist is a pure inertial oscillation of the entire fluid. This motion involves a balance between the local acceleration and the Coriolis force. No such simple type of oscillation exists in the present case because there are horizontal gradients and cellular boundaries. However, an oscillation which involves a partial balance between the local acceleration and the Coriolis force is possible and does take place.

In steady convection the rotational constraint balances much of the horizontal temperature gradient, i.e. the force which releases potential energy. Now consider what happens when a transient motion can exist at the onset of convection. A transient motion means that part of the Coriolis force can be balanced by the local acceleration so that less of the rotational constraint is available to offset the horizontal temperature gradient. Consequently, the cell is distorted (shrunk) less by the rotation and there is less dissipation associated with the somewhat larger cell. Convection can, therefore, be maintained for a smaller imposed temperature difference (smaller Rayleigh number).

In the balance of processes we note that a time-dependent temperature field involves a perturbation temperature which is out of phase with the vertical velocity and hence is less efficient than the steady motion for convecting heat upward. It is clear that overstable motions are preferred only when the effects of these out-of-phase temperature fluctuations are smaller than the effects of the time-dependent motions in the dynamical processes because the latter enhance convection by offsetting the constraining force of rotation. The Prandtl number measures the relative roles of viscosity and conductivity. When σ is small, viscous forces are relatively less important than diffusive processes and time-dependent motions are more important in the dynamical balance (where they are helpful) than in the thermal balance (where they are inefficient). Hence the onset of convection as overstable motions for smaller σ.

The same kind of qualitative argument can be used in connexion with inertial processes. Thus it is possible that motions with finite amplitude may exist at subcritical values of the Rayleigh number because inertial processes may balance the constraining effect of rotation. In I we have seen that such is the case for motions of small but finite amplitude. We investigate here the effect of large amplitude.

4. Results with a limited representation

It is instructive to look at a highly truncated representation of the various fields and to deduce certain general physical results with a minimum amount of mathematical analysis. Then, using these results as a guide, we shall proceed with the fully non-linear problem in a later paper.

The stability problem has a steady solution whose form is given by the expressions (3.1) for ψ, T and v. The first effect of non-linearity is to distort the temperature field through the interaction of ψ and T and the zonal velocity field through the interaction of ψ and v. The distortion of the temperature field will correspond to a change in the horizontal mean, i.e. a component of the form $\sin 2\pi z$ will be generated. Similarly, the zonal velocity field will be distorted by a component of the form $\sin 2\pi\alpha x$. (A change in the horizontal mean of v corresponds to a translation of the co-ordinate system and is of no interest.)

Therefore, the minimal system which describes finite-amplitude convection is given by

$$\psi = A(t)\sin\pi\alpha x\sin\pi z, \tag{4.1a}$$

$$T = B(t)\cos\pi\alpha x\sin\pi z + C(t)\sin 2\pi z, \tag{4.1b}$$

$$v = D(t)\sin\pi\alpha x\cos\pi z + E(t)\sin 2\pi\alpha x, \tag{4.1c}$$

where the amplitudes A, B, C, D, E can generally be functions of time, t, and are to be determined by the dynamics of the system. If expressions (4.1) are substituted into equations (2.15), (2.16) and (2.17) and if we equate corresponding coefficients of $\sin\pi\alpha x\sin\pi z$, $\cos\pi\alpha x\sin\pi z$, etc., we deduce the following set of equations as the deterministic set for the amplitudes

$$\dot{A} = \pi^4(\alpha^2+1)^2A + \pi\alpha RB - \pi\mathscr{T}D, \tag{4.2a}$$

$$\dot{B} = -\pi^2\alpha AC - \pi\alpha A - \pi^2(\alpha^2+1)B, \tag{4.2b}$$

$$\dot{C} = \tfrac{1}{2}\pi^2\alpha AB - 4\pi^2C, \tag{4.2c}$$

$$\dot{D} = \pi^2\alpha AE - \pi\sigma\mathscr{T}A - \sigma\pi^2(\alpha^2+1)D, \tag{4.2d}$$

$$\dot{E} = -\tfrac{1}{2}\pi^2\alpha AD - 4\sigma\pi^2\alpha^2E, \tag{4.2e}$$

where the dot over corresponds to a time derivative.

This set of non-linear ordinary differential equations is too complicated to solve for the general time-dependent fields. However, we can look at the steady-state solutions to the system. It turns out that this information is very useful to have because it predicts that a finite-amplitude *steady* solution to the system is possible for subcritical values of the Rayleigh number and that the minimum values of R for which a steady solution is possible lies below the critical values for instability to either a steady infinitesimal perturbation or an overstable infinitesimal perturbation.

Setting the left-hand sides of equations (4.2) equal to zero we have

$$\pi^4(\alpha^2+1)A+\pi\alpha RB-\pi\mathscr{T}D = 0, \tag{4.3a}$$

$$\pi^2\alpha AC+\pi\alpha A+\pi^2(\alpha^2+1)B = 0, \tag{4.3b}$$

$$\tfrac{1}{2}\pi^2\alpha AB-4\pi^2C = 0, \tag{4.3c}$$

$$\pi^2\alpha AE-\pi\sigma\mathscr{T}A-\sigma\pi^2(\alpha^2+1)D = 0, \tag{4.3d}$$

$$\tfrac{1}{2}\pi^2\alpha AD+4\sigma\pi^2\alpha^2E = 0. \tag{4.3e}$$

Elimination of all amplitudes except for A is accomplished by the deduced relations

$$C = \tfrac{1}{8}\alpha AB, \tag{4.4a}$$

$$E = -AD/8\sigma\alpha, \tag{4.4b}$$

$$B = -\pi\alpha A/\pi^2[\alpha^2+1+\tfrac{1}{8}\alpha^2A^2], \tag{4.4c}$$

$$D = -\pi\mathscr{T}A/\pi^2[\alpha^2+1+A^2/8\sigma^2]. \tag{4.4d}$$

Substituting (4.4c) and (4.4d) into (4.3a) yields the equation (after some algebraic simplification)

$$A\left\{\frac{\alpha^2}{\sigma^2}\pi^4(\alpha^2+1)^2\left(\frac{A^2}{8}\right)^2+\left[\frac{\pi^4(\alpha^2+1)^3}{\sigma^2}+\alpha^4R_c-\frac{\alpha^2}{\sigma^2}R\right]\frac{A^2}{8}+(\alpha^2+1)\alpha^2(R_c-R)\right\} = 0. \tag{4.5}$$

The solution $A = 0$ corresponds to pure conduction, which we know to be a possible solution though it is unstable when R is sufficiently large. The remaining solutions are given by

$$\frac{A^2}{8} = \left[\frac{\alpha^2}{\sigma^2}(R-R_c)-\alpha^4R_c+\frac{\mathscr{T}^2}{\sigma^2}\pm\left\{\left[\frac{\alpha^2}{\sigma^2}(R-R_c)-\alpha^4R_c+\frac{\mathscr{T}^2}{\sigma^2}\right]^2+\frac{4\alpha^4}{\sigma^2}(\alpha^2R_c-\mathscr{T}^2)(R-R_c)\right\}^{\frac{1}{2}}\right]\bigg/\frac{2\alpha^2}{\sigma^2}\pi^4(\alpha^2+1)^2. \tag{4.6}$$

Only the solution with the positive sign in front of the radical is admissible since otherwise A^2 is negative, i.e. the amplitude of the stream function is imaginary.

Consider the case where finite solutions exist for $R < R_c$. The minimum value of R for which solutions exist is that value of R which makes the radical vanish provided that the first term on the right-hand side of (4.6) be non-negative. The radical vanishes provided that

$$R = [\{(1/\alpha^2-\sigma^2)(1+\alpha^2)^3\pi^4\}^{\frac{1}{2}}+\mathscr{S}]^2. \tag{4.7}$$

(The alternative possibility of a negative sign in front of $\mathscr{S}$ causes A^2 to be negative, hence is not admissible.) With this value of R, amplitudes are real provided that the first term on the right-hand side of (4.6) be non-negative, or equivalently

$$\mathscr{S}^2 \geqslant \sigma^4(1+\alpha^2)^3\pi^4/(1/\alpha^2-\sigma^2). \tag{4.8}$$

Conditions (4.7) and (4.8) are meaningful only when

$$\sigma^2 < \alpha^{-2}. \tag{4.9}$$

And finally we note that $R_f^{\min}$, the minimum value of R at which a steady finite-amplitude solution can exist, occurs for

$$\alpha^2 = [1 \pm (1-3\sigma^2)^{\frac{1}{2}}]/3\sigma^2. \tag{4.10}$$

For small σ (4.10) (with the negative sign) becomes $\alpha^2 \sim \frac{1}{2}$ and

$$R_f^{\min} \sim [\{(2-\sigma^2)\tfrac{27}{8}\pi^4\}^{\frac{1}{2}} + \sigma\mathscr{T}]^{\frac{1}{2}}. \tag{4.11}$$

For mercury $\sigma = 0{\cdot}025$ so that (4.11) is satisfied. We note that

$$R_f^{\min} \sim [(\tfrac{27}{4}\pi^4)^{\frac{1}{2}} + \mathscr{S}]^2 \tag{4.12}$$

with an asymptotic (for large $\mathscr{S}$) value of $R_f^{\min} \to \mathscr{S}^2$. From the results of the stability problem we know that asymptotically $R_0^{\min}$ is achieved for large values of α^2 and that it behaves as $\mathscr{S}^{\frac{4}{3}}$. $R_f^{\min}$ as we have seen grows as $\mathscr{S}^2$. Thus in the asymptotic range instability cannot manifest itself in the form of finite-amplitude steady motions. The reason for this is (as reported in I) that finite amplitude motions can occur in a subcritical range of R only as long as they can reduce the constraint of rotation. When the effect of the latter is large, the motion must have larger amplitudes in order to offset the constraint, Greater amplitudes require more release of potential energy, which in turn requires a larger value of R. Hence, when $\mathscr{T}^2$ becomes large enough, in order to offset the constraint the motion must have an amplitude which cannot be achieved for $R < R_0$. However, as long as $\mathscr{S}$ is not too large, finite-amplitude steady motions can exist for values of R less than $R_0^{\min}$. This result is really a significant one considering the fact that $R_0^{\min}$ is considerably smaller than $R_c^{\min}$, i.e. *infinitesimal steady* convection requires a much higher value of the Rayleigh number than does infinitesimal *overstable* convection. These results are summarized in figure 1 where we show the ranges of preferred types of motion for mercury, i.e. $\sigma = 0{\cdot}025$.†

The present analysis is, of course, restricted to a severely truncated representation. This fact is crucial especially since finite-amplitude motions which occur for values of R far below the critical value may require a representation quite different from that which is adequate to describe motions of small finite amplitude near the critical Rayleigh number. We should also keep in mind the fact that finite-amplitude *unsteady* motions at subcritical values of the Rayleigh number may be possible.

An additional important point to note here is that the preferred *scale* of the finite-amplitude motions is larger (α^2 is smaller) than the corresponding scale for infinitesimal overstable motions. Thus two significant qualitative results should be observed in an experiment with mercury. The motions should be steady (rather than overstable) for values of rotation corresponding to moderate $\mathscr{S}^2$ and the scale of the motion should be comparable to that which occurs for non-rotating convection.

† The curves for $R_0^{\min}$ and $R_f^{\min}$ are essentially unchanged for smaller values of σ. However, $R_c^{\min}$ is shifted upward and to the left as σ is decreased.

For fluids with Prandtl number greater than α^{-1},† no finite-amplitude motions should exist below the critical Rayleigh number. For these fluids equation (4.6) gives the amplitudes of the motions for the limited representation. In a later paper we shall explore the present result and the changes brought about by a larger representation. Also, by means of numerical integrations of the determining equations we shall answer the questions brought up in this section. A preliminary result with a larger representation confirms the existence of steady motions at subcritical Rayleigh numbers.

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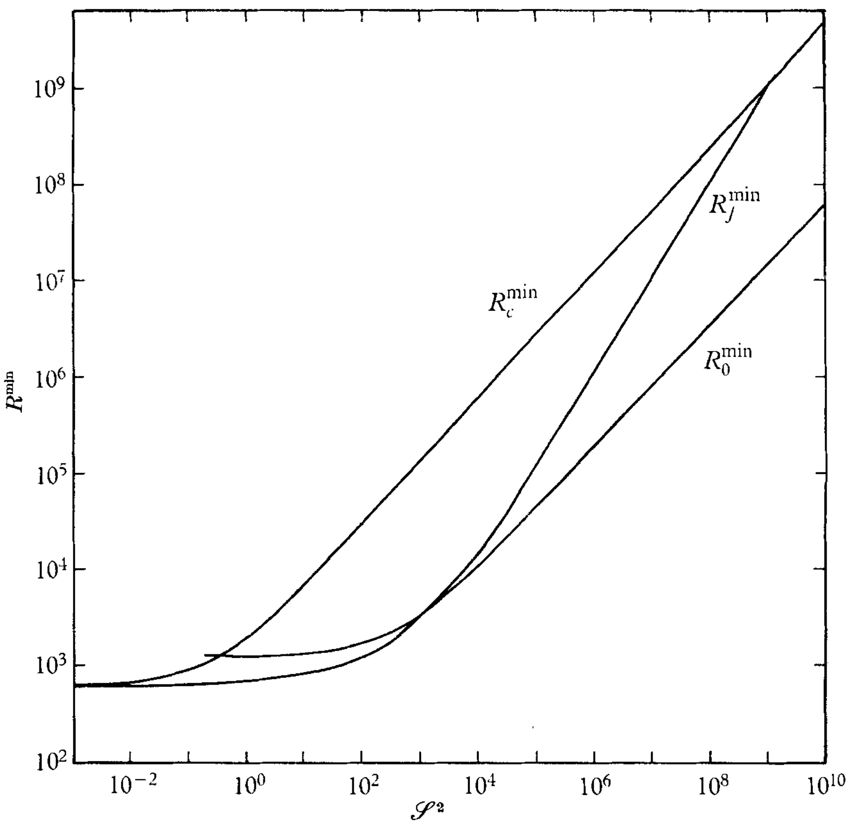

FIGURE 1. Curves of $R_c^{\min}$, $R_0^{\min}$ and $R_J^{\min}$ are shown as functions of the modified Taylor number $\mathscr{S}^2$. For the numerical values the value $\sigma = 0{\cdot}025$ has been taken. Finite-amplitude steady motions should exist for subcritical R for the range $\mathscr{S}^2 < 10^3$. For $\mathscr{S}^2 > 10^3$ it is possible that subcritical overstable motions exist.

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† It should be noted that this criterion implies that finite amplitude instability can occur for fluids with $\sigma > 1$, i.e., where overstability is not possible.